

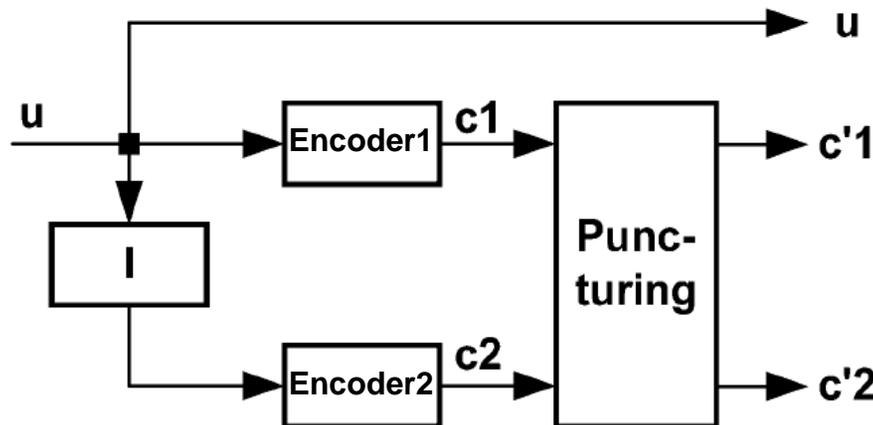


# Tail-Biting Encoding for 3GPP LTE Turbo Codes of Arbitrary Number of Information Bits

# Outline

- What is tail-biting termination of turbo coding
- Matrix representation of tail-biting termination for convolutional encoder
- When does tail-biting fails
- Tail-biting termination for arbitrary number of information symbols
- An application on parallel turbo decoding

# Tail-biting Termination for Turbo Coding



Input sequence to one encoder:  $u_0, u_1, \dots, u_{k-1}$

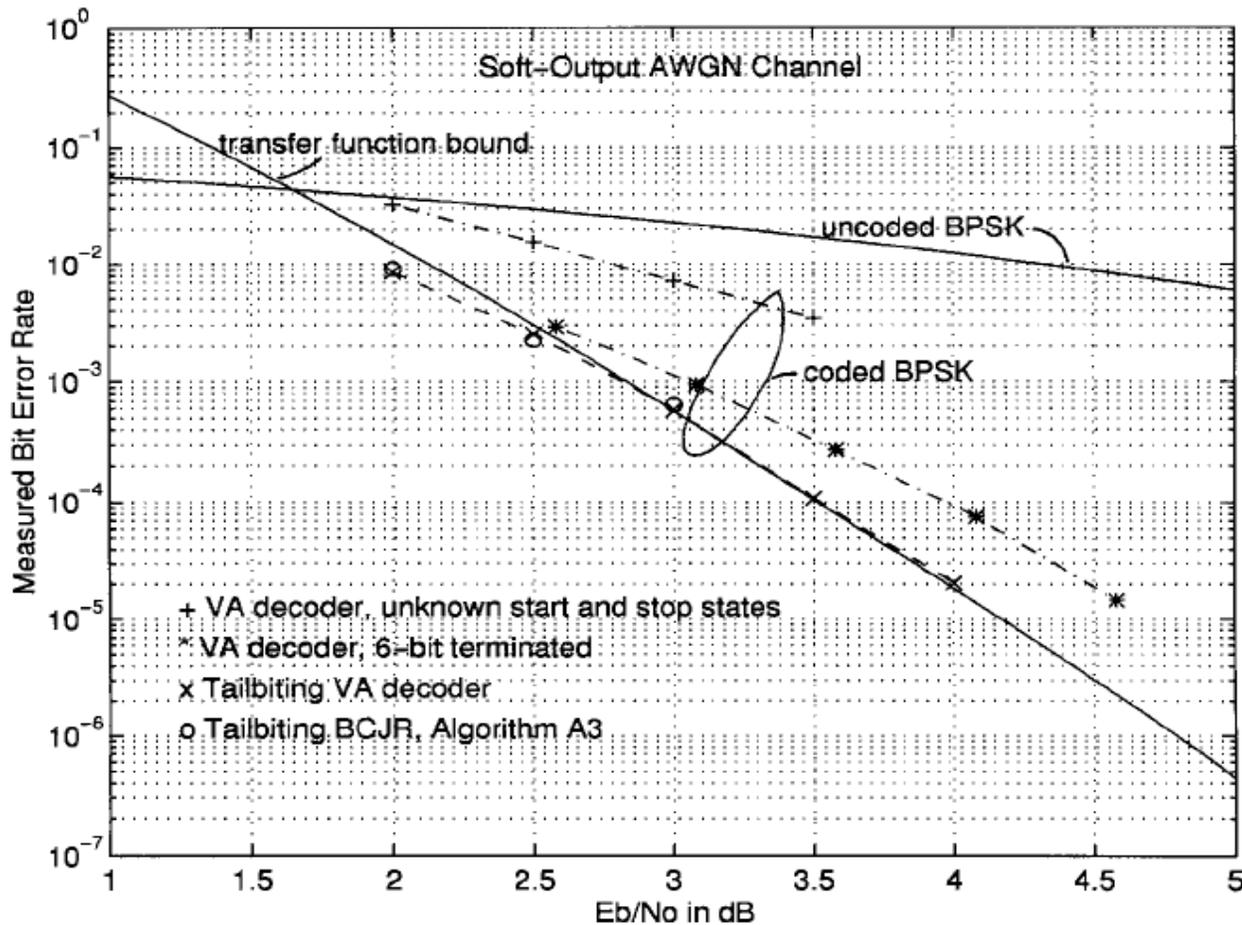
States sequence of one encoder:  $S_0, S_1, \dots, S_k$

Forward and backward decoding algorithms of convolutional codes depends on  $S_0$  and  $S_1$

Tail-biting termination  $\iff S_0 = S_k$ .

Tail-biting termination guarantees uniform protection with no extra overhead

# Performance Comparison [1]



Bit-error rates for four decoders working with the  $R = 1/2$  memory 6 code (554, 744)

# State-Space Realization of Convolutional Encoder [2]

Rate  $k_0/n_0$  convolutional encoder

$$\text{Input data } \mathbf{u} = (u_0, \dots, u_{N-1}), \quad u_i = (u_{i,0}, \dots, u_{i,k_0-1})$$

$$\text{Output data } \mathbf{x} = (x_0, \dots, x_{N-1}), \quad x_i = (x_{i,0}, \dots, x_{i,n_0-1})$$

$$\text{State at time } t \quad (s_{m-1}^{(t)}, \dots, s_0^{(t)})$$

There exist matrices (A,B,C,D), A:  $m \times m$ , B:  $m \times k_0$ , C:  $n_0 \times m$ , D:  $k_0 \times n_0$

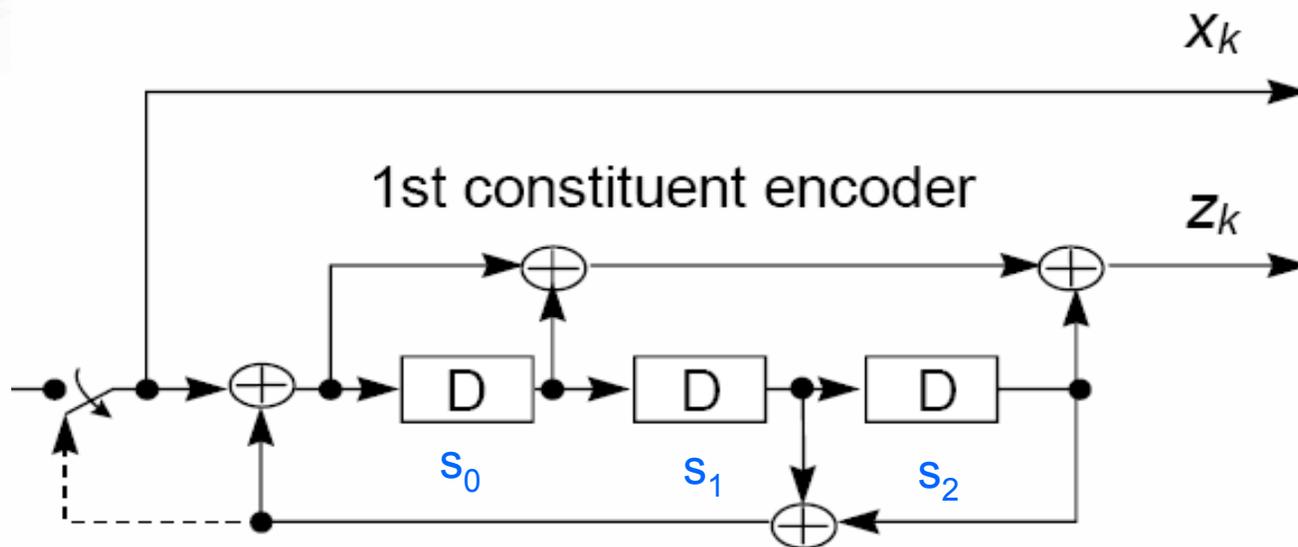
$$(s_{m-1}^{(t)}, \dots, s_0^{(t)})^T = A(s_{m-1}^{(t-1)}, \dots, s_0^{(t-1)})^T + Bu_t^T$$

$$x_t^T = C(s_{m-1}^{(t-1)}, \dots, s_0^{(t-1)})^T + Du_t^T$$

(A,B,C,D): state-space realization of the convolutional code

m: degree of the realization (A,B,C,D) ( $2^m$  is the # states)

# State-Space realization of Convolutional Encoder for Turbo Code Rel.6 [3]



$$s_0^t = s_2^{t-1} + s_1^{t-1} + u_t, s_1^t = s_0^{t-1}, s_2^t = s_1^{t-1}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Degree = 3

# Algebraic Description of Tail-biting Encoder

Initial state:  $(s_0^{(0)}, \dots, s_{m-1}^{(0)})^T$

Final state:  $(s_0^{(N)}, \dots, s_{m-1}^{(N)})^T = A^N (s_0^{(0)}, \dots, s_{m-1}^{(0)})^T + \sum_{r=0}^{N-1} A^{N-1-r} B u_r^T$

Tail-biting:  $(s_0^{(N)}, \dots, s_{m-1}^{(N)}) = (s_0^{(0)}, \dots, s_{m-1}^{(0)})$

$$\implies (s_0^{(0)}, \dots, s_{m-1}^{(0)})^T = A^N (s_0^{(0)}, \dots, s_{m-1}^{(0)})^T + \sum_{r=0}^{N-1} A^{N-1-r} B u_r^T$$

The encoder with state-space realization (A,B,C,D) is tail-biting if and only if

$$(A^N + I_m)(s_0^{(0)}, \dots, s_{m-1}^{(0)})^T = \sum_{r=0}^{N-1} A^{N-1-r} B u_r^T$$

has solution for all possible input  $\mathbf{u}$

# Necessary and Sufficient Condition on Tail-biting Encoder

Minimal degree  $m$  of a convolutional encoder  $(A,B,C,D)$ :

- 1)  $m$  is the degree of  $(A,B,C,D)$
- 2) The encoder cannot be realized by  $(A',B',C',D')$  of degree  $< m$

Degree 3 encoder:  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, C, D$

can be reduced to degree 2 encoder  $A' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C', D$

# Necessary and Sufficient Condition on Tail-biting Encoder

**Theorem 1** Convolutional encoder with minimal degree  $m$  state-space realization  $(A,B,C,D)$  is tail-biting for the information sequence of size  $N \geq m$

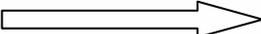
If and only if  $(A^N + I_m)$  is invertible

The encoder in Rel.6 turbo code  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$   $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$   $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$   $D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  has minimal degree

$A^7 = I_3$ , but  $A + I_3, A^2 + I_3, A^3 + I_3, A^4 + I_3, A^5 + I_3, A^6 + I_3$  are invertible

$$\Rightarrow A^N + I_3 = \begin{cases} \textit{invertible} & N \bmod 7 \neq 0 \\ \textit{non-invertible} & N \bmod 7 = 0 \end{cases}$$

# All Turbo Encoder Fails Tail-biting at Some Information Size

Turbo encoder  Recursive convolutional encoders

**Theorem 2** For any recursive convolution encoder of minimal degree  $m$  there exists a positive integer  $P$  such that this encoder gives no tail-biting termination for some information sequences of size  $N=nP$  ( $n>0$ ).

The maximal of  $P$  for a convolutional encoder  $(A,B,C,D)$  of degree  $m$  is  $2^m-1$

- #states = 4:  $P=3$
- #states = 8:  $P=7$
- #states = 16:  $P=15$
- #states = 32:  $P=31$

# 8 States Convolutional Encoders

Similar matrices:  $m \times m$  matrices  $A_1$  and  $A_2$  are similar iff there exists an Invertible matrix  $S$  such that

$$A_1 = SA_2S^{-1}$$

Encoder with state-space realization  $(A,B,C,D)$  and  
Encoder with state-space realization  $(SAS^{-1},SB,CS^{-1},D)$   
are equivalent (i.e. have the same generator matrix)

8 states  $(A,B,C,D)$  have 14 different equivalent classes

3 types of classes: I) invertible; II) nilpotent ( $A^n=0$ ); III) either 1) and 2)

# 8 States Convolutional Encoders

## Classes II: Nilpotent, 3 representatives

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

These matrices are *none recursive* and can not be considered as a constituent encoder of turbo code

# 8 States Convolutional Encoders

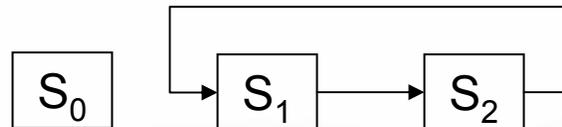
## Classes III: Non-invertible and non-nilpotent , 5 representatives

Case 1) 4 representatives

$$A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Encoder with these 4 state matrices will give a *disconnected memory*.  
The encoder with disconnected memory will not give the best  $d_2$  needed by turbo codes .

Example  $A_5 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$



# 8 States Convolutional Encoders

## Classes III: Non-invertible and non-nilpotent , 5 representatives

Case 2) one representatives

$$A_8 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow A_8^t = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad t > 1 \rightarrow A_8^t + I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ none invertible}$$

( $A_8, B, C, D$ ) either is not tail-biting for the information sequence size  $> 2$   
or it is not minimal

Example: a) take  $B = [1 \ 1 \ 1]^T \rightarrow B, BA = [0 \ 1 \ 0]^T$  and  $BA^2 = [1 \ 0 \ 1]^T$  are linear dependent  $\rightarrow$  no tail-biting for some sequence of size  $> 2$   
b) take  $B = [1 \ 1 \ 0]^T \rightarrow (A, B, C, D)$  can be reduced to a degree 2 encoder ( $A', B', C', D$ ) with

$$A' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

# 8 States Convolutional Encoders

**Classes I: invertible, 6 representatives**

Case 1) 3 representatives

$$A_9 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A_{10} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_{11} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



*disconnected memory encoders*

# 8 States Convolutional Encoders

**Classes I: invertible, 6 representatives**

Case 2) 1 representatives

$$A_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow A_{12}^4 = I_3, A_{12}^2 + I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_{12}^3 + I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{12}^N + I_3 = \begin{cases} \textit{invertible} & N = 1 \\ \textit{non-invertible} & N > 1 \end{cases}$$

No tail-biting for all possible size  $> 1$

# 8 States Convolutional Encoders

**Classes I: invertible, 6 representatives**

Case 3) 2 representatives

$$A_{13} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow A_{13}^7 = I, \quad \text{Use in Rel.6 rate 1/3 turbo code}$$

$$A_{13}^1 + I_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A_{13}^2 + I_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_{13}^3 + I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

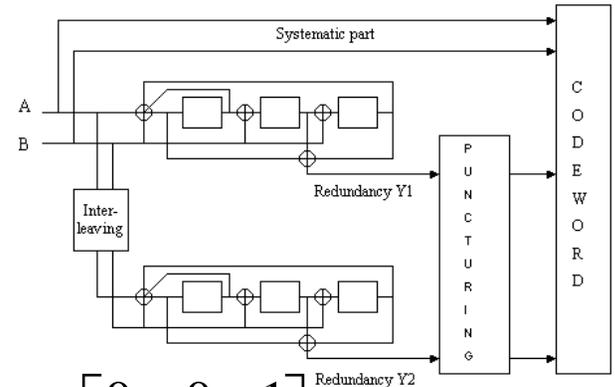
$$A_{13}^4 + I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_{13}^5 + I_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_{13}^6 + I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

# 8 States Convolutional Encoders

**Classes I: invertible, 6 representatives**

Case 3) 2 representatives

$$A_{14} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow A_{14}^7 = I_3, \quad \text{Duo-binary turbo code}$$



$$A_{14}^1 + I_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_{14}^2 + I_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad A_{14}^3 + I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$A_{14}^4 + I_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_{14}^5 + I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_{14}^6 + I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

# 8 States Convolutional Encoders

Conclusion: only two none equivalent classes can be used for tail-biting encoder  
one is used in Rel.6  
one is used in duo-binary turbo code

# Tail-Biting Termination for Arbitrary Number of Information symbols

(A,B,C,D): state space realization of the 8 states convolutional encoder with  $A^7=I_3$

I: Pre-compute the followings states for  $i=1,2,3,4,5,6$

$$S_{i,1} = (A^i + I_3) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, S_{i,2} = (A^i + I_3) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, S_{i,3} = (A^i + I_3) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

$$S_{i,4} = (A^i + I_3) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, S_{i,5} = (A^i + I_3) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, S_{i,6} = (A^i + I_3) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, S_{i,7} = (A^i + I_3) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

II: Store the above 42 states as a look-up-table  $L(i, b_{(2)}) = S_{i,b}$ , where  $b=1,2,3,4,5,6,7$  and  $b_{(2)}$  is the 3 bits binary representation of  $b$ . Moreover, let  $L(i,0)=0$  state.

# Tail-Biting Termination for Arbitrary Number of Information symbols

Tail-biting encoding method for information block size =  $k$

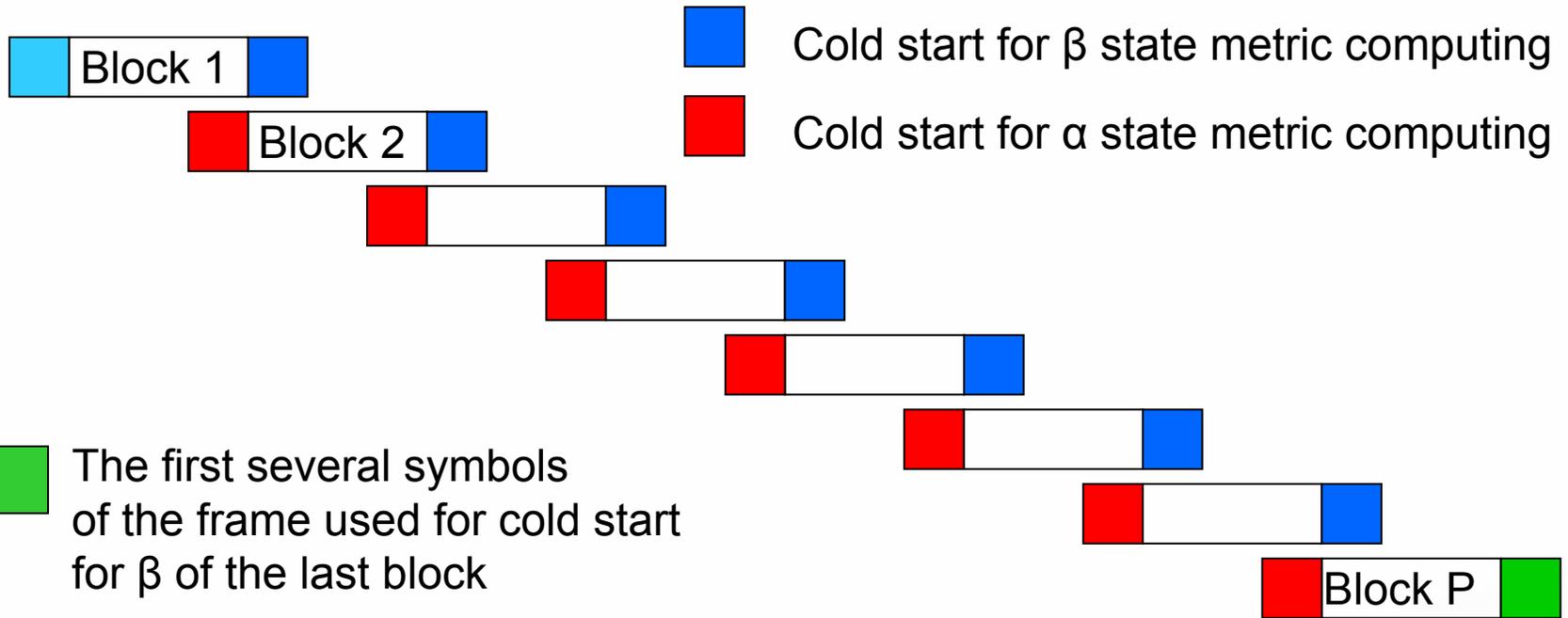
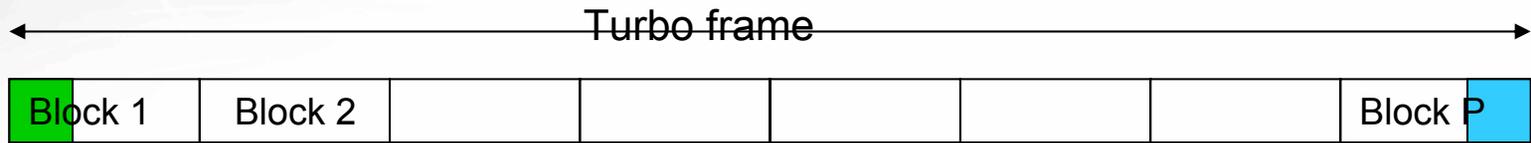
Let  $u_0, u_1, \dots, u_{k-1}$  be the information symbols.

1. Let  $m = (k \bmod 7)$ . If  $m=0$ , pad one more symbol  $u_k=0$  and let  $N=k+1$  and  $M=1$ , otherwise let  $N=k$  and  $M=m$ .
2. With 0 state encoding information symbols  $u_0, u_1, \dots, u_{N-1}$  to find the final state  $S_{\text{final}}$  (do not store the encoded symbols). then use Look-up table to find the initial state  $S_0 = L(M, S_{\text{final}})$ .
3. Use  $S_0$  as initial state to encode  $u_0, u_1, \dots, u_{N-1}$

# Advantage of our proposal

- **Flexible:** No limitation of the size of information symbols sequence
- **Less overhead:** at most 1 bit overhead. Better than adding  $2^m$  overhead termination bits for an  $2^m$  state convolutional code used in 3GPP Rel.6 turbo codes and other turbo codes
- **Better performance:** all the bits are interleaved while the 6 termination bits are not interleaved.

# Parallel Decoding for Tail-Biting encoded symbols



 The first several symbols of the frame used for cold start for  $\beta$  of the last block

 The last several symbols of the frame used for cold start for  $\alpha$  of the first block

# References

- [1] J. B. Anderson and S. M. Hladik, "Tailbiting MAP Decoders," *IEEE JSAC*, VOL. 16, NO. 2, FEBRUARY 1998
- [2] R.J. McEliece. The Algebraic Theory of Convolutional Codes. In *Handbook of Coding Theory*, R. Brualdi, W.C. Human and V. Pless (eds.). Elsevier Science Publishers, Amsterdam, The Netherlands, 1998
- [3] 3GPP TS 25.212 V6.8.0 (2006-06)
- [4] H. Gluesing-Luerssen and G. Schneider, "State space realizations and monomial equivalence for convolutional codes", *arXiv:cs.IT/0603049*, Mar. 2006
- [5] C. Berrou, A. Glavieux and P. Thitimajshima, "Near Shannon limit error-correcting coding and decoding: turbo codes", *Proc. of IEEE ICC '93, Geneva*, pp. 1064-1070, May 1993.
- [6] Duo-Binary Turbo Codes for Evolved UTRA, *3GPP TSG RAN WG1#46 (R1-061973)*